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THE DISTANCE BETWEEN A POINT AND AN ELLIPSOID

V. Kucher

October 1977

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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: BRL Report No. 2024 (14) BRL-2024
2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): (6) THE DISTANCE BETWEEN A POINT AND AN ELLIPSOID
5. TYPE OF REPORT & PERIOD COVERED: (9) Final rept.,
6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): (10) V. Kucher
8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS: US Army Ballistic Research Laboratory, Aberdeen Proving Ground, MD 21005
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: (11) 1W161102AH43
11. CONTROLLING OFFICE NAME AND ADDRESS: US Army Materiel Development & Readiness Command, 5001 Eisenhower Avenue, Alexandria, VA 22333
12. REPORT DATE: (11) OCTOBER 1977
13. NUMBER OF PAGES: 23 (12) 22 p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
15. SECURITY CLASS. (of this report): Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report): Approved for public release; distribution unlimited.

DDC RECEIVED FEB 23 1978 B

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)
18. SUPPLEMENTARY NOTES
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Calculus of Variations
Lagrange Multipliers
Optimization
Kill Probability

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The method of the calculus of variations is used to determine the distance from any point, not in a coordinate plane, to the surface of an ellipsoid. The distance from any point in the coordinate plane to the surface of an ellipsoid is formulated by the method of Lagrange multipliers. A computer program has been developed to compute this distance.

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INTRODUCTION

A missile can kill a target in several ways, for example, by an internal or external blast from the warhead and by fragments from the warhead.[1] A mathematical model for determining the single-shot kill probability requires that the path of the fragments, the shape of the blast envelope, the shape of the target, etc. be expressed mathematically. J. von Neumann suggested that a set of ellipsoids could be used to define the blast envelope and the shape of the target.[2] For swept-back configurations of the target, a set of cardiaci can be used to approximate the target mathematically.[3] The initiation of certain fuzes on the missile require knowledge of the distance between the target and the missile's fuze. Mathematically, this is the distance between the fuze and the nearest ellipsoid describing the target.

METHOD OF THE CALCULUS OF VARIATIONS

We wish to find the shortest distance, d, between a fixed point, $P(x_0, y_0, z_0)$, and the surface, $G(x,y,z) = 0$. $Q(x_1, y_1, z_1)$ is defined to be a point on this surface such that the distance from P to Q is a minimum. See Figure 1.

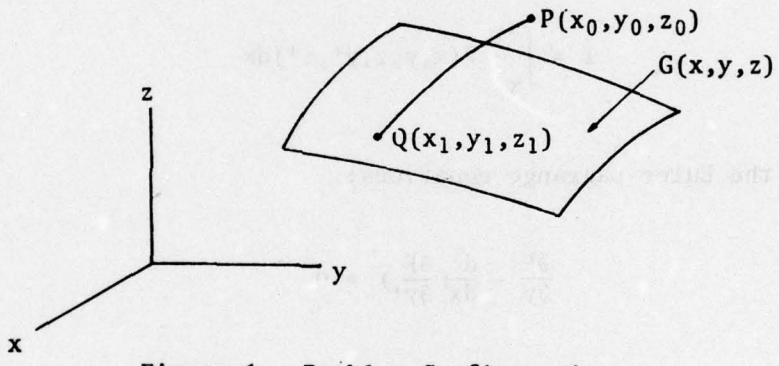

Figure 1. Problem Configuration

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1. A. Stiegler, "A Mathematical Formulation for ORDVAC Computation of the Single-Shot Kill Probabilities of a General Missile Versus a General Aircraft," Ballistic Research Laboratories Memorandum Report No. 1306, November 1960. (AD #249957)

2. M. Juncosa and D. Young, "A Mathematical Formulation for ORDVAC Computation of the Probability of a Kill of an Airplane by a Missile," Ballistic Research Laboratories Report No. 867, 1953. (AD #17267)

3. V. Kucher, "Mathematical Approximation of Sweptback Configurations by Cardiaci," Ballistic Research Laboratories Memorandum Report No. 1793, October 1966. (AD #806663)

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We shall consider the surface to be an ellipsoid such that

$$G(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \quad a,b,c > 0. \quad (1)$$

Any ellipsoid that is not expressed in standard form can, by translation and rotation transformations, be written in the form of Equation 1. Naturally, these transformations would also be applied to the fixed point in space.

The length of any curve extending from point P to point Q is given as

$$S = \int_{x_0}^{x_1} [1 + (y')^2 + (z')^2]^{\frac{1}{2}} dx. \quad (2)$$

A procedure for minimizing an integral of the form[4]

$$I = \int_{x_0}^{x_1} F(x,y,z,y',z')dx \quad (3)$$

is to apply the Euler-Lagrange equations:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad (4)$$

and

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0. \quad (5)$$

---

4. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. I, New York, Interscience Publishers, 1953.

From Equation (2) we see that for our case

$$F = [1 + (y')^2 + (z')^2]^{\frac{1}{2}}. \tag{6}$$

The Euler-Lagrange equations take the form

$$[1 + (z')^2]\, y'' - y'z'z'' = 0 \tag{7}$$

and

$$[1 + (y')^2]\, z'' - z'y'y'' = 0. \tag{8}$$

The solution to the simultaneous equations [(7) and (8)] is

$$y = C_1x + C_2, \tag{9}$$

$$z = C_3x + C_4, \tag{10}$$

where C_1, C_2, C_3, and C_4 are constants of integration. As expected, we have found that the shortest curve between points P and Q is a straight line.

Applying the conditions at point P that $x = x_0$, $y = y_0$, and $z = z_0$, we have

$$y_0 = C_1x_0 + C_2, \tag{11}$$

$$z_0 = C_3x_0 + C_4. \tag{12}$$

The next step in our procedure to minimize the distance between points P and Q is to apply the transversality conditions:

$$\left(y'\frac{\partial F}{\partial y'} + z'\frac{\partial F}{\partial z'} + \nu\frac{\partial G}{\partial x} - F\right)\Big|_Q = 0, \tag{13}$$

$$\left(\frac{\partial F}{\partial y'} - \nu\frac{\partial G}{\partial y}\right)\Big|_Q = 0, \tag{14}$$

$$\left(\frac{\partial F}{\partial z'} - \nu\frac{\partial G}{\partial z}\right)\Big|_Q = 0, \tag{15}$$

where ν is a Lagrange multiplier.

Equations (13), (14), and (15) yield, respectively,

$$2\nu x_1[1 + (y_1')^2 + (z_1')^2]^{\frac{1}{2}} - a^2 = 0, \tag{16}$$

$$2\nu y_1[1 + (y_1')^2 + (z_1')^2]^{\frac{1}{2}} - b^2 y_1' = 0, \tag{17}$$

$$2\nu z_1[1 + (y_1')^2 + (z_1')^2]^{\frac{1}{2}} - c^2 z_1' = 0. \tag{18}$$

Equations (16), (17), and (18) give

$$y_1' = \frac{a^2 y_1}{b^2 x_1}, \quad x_1 \neq 0, \tag{19}$$

$$z_1' = \frac{a^2 z_1}{c^2 x_1}, \quad x_1 \neq 0. \tag{20}$$

The slope of the line given by Equations (9) and (10) is provided by Equations (19) and (20). Thus

$$C_1 = \frac{a^2 y_1}{b^2 x_1} \text{ and } C_3 = \frac{a^2 z_1}{c^2 x_1}. \qquad (21) - (22)$$

Note that the slope of the surface of the ellipsoid at point Q can be determined to be the negative reciprocal of the slope given by Equations (19) and (20); consequently, the line is perpendicular to the surface of the ellipsoid. Thus transversality coincides with orthogonality.

Substituting Equations (21) and (22) into Equations (11) and (12), we find that

$$C_2 = y_0 - \frac{a^2 y_1}{b^2 x_1} x_0, \qquad (23)$$

$$C_4 = z_0 - \frac{a^2 z_1}{c^2 x_1} x_0. \qquad (24)$$

Substituting the four constants of integration into Equations (9) and (10), we obtain the equation of the shortest curve between points P and Q:

$$y = \frac{a^2 y_1}{b^2 x_1} (x - x_0) + y_0, \qquad (25)$$

$$z = \frac{a^2 z_1}{c^2 x_1} (x - x_0) + z_0. \qquad (26)$$

Evaluating Equations (25) and (26) at Q, we find that

$$y_1 = \frac{b^2 y_0 x_1}{b^2 x_1 - a^2(x_1 - x_0)}, \tag{27}$$

$$z_1 = \frac{c^2 z_0 x_1}{c^2 x_1 - a^2(x_1 - x_0)}. \tag{28}$$

If we evaluate Equation (1) at Q and substitute Equations (27) and (28) into the resulting equation, we eliminate y_1 and z_1 and obtain a sixth-degree polynomial in x_1:

$$K_0 x_1^6 + K_1 x_1^5 + K_2 x_1^4 + K_3 x_1^3 + K_4 x_1^2 + K_5 x_1 + K_6 = 0 \tag{29}$$

with coefficients given by

$$K_0 = r^2 s^2,$$

$$K_1 = 2rskt,$$

$$K_2 = k^2 u + a^2(gs^2 + hr^2 - r^2 s^2),$$

$$K_3 = 2k[k^2 t + a^2(gs + hr - rst)],$$

$$K_4 = k^2[k^2 + a^2(g + h - u)],$$

$$K_5 = -2a^2 k^3 t,$$

$$K_6 = -a^2 k^4,$$

where

$$r = b^2 - a^2,$$

$$s = c^2 - a^2,$$

$$g = y_0^2 b^2,$$

$$h = z_0^2 c^2,$$

$$k = x_0 a^2,$$

$$t = r + s$$

$$u = r^2 + 4rs + s^2.$$

The reader may use his favorite method or computer routine to find the roots of Equation (29). Since K_6 is negative and K_0 is positive, we are assured that at least one positive real root and one negative real root exist. The fact that $|x_1| \leq a$ is useful in the determination of the roots which have meaning in this problem.

If point $P(x_0,y_0,z_0)$ does not lie in any of the coordinate planes, values of y_1 and z_1, corresponding to each of the real roots ($|x_1| \leq a$) of Equation (29), are obtained from Equations (27) and (28). Point $Q(x_1,y_1,z_1)$ is found by determining the minimum value of d for each triplet (x_1,y_1,z_1) obtained, where

$$d = [(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]^{\frac{1}{2}}. \qquad (30)$$

As an example of the above procedure, let (a,b,c) = (4,3,2) and (x_0,y_0,z_0) = (1,1,1). The roots of Equation (29) for this example were determined by an available computer routine based on the procedure developed by Bareiss and Hamelick.[5] However, other techniques, such as the Newton-Raphson method, should be adequate for this purpose. A summary of the calculations is listed:

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5. *E. Bareiss and R. Hamelink, "RSSR Routine, A Root-Squaring and Subresultant Procedure for Finding Zeros of Real Polynomials," Argonne National Laboratory Report No. ANL-6987, October 1965.*

Root	x_1	y_1	z_1	d	
1	1.118,521,8	1.232,101,4	1.735,663,8	0.780,461,2	(minimum)
2	-3.821,653,6	-0.804,529,0	-0.247,116,9	5.297,166,1	

The procedure, which was developed for determining Q and d, may require some modification if point P lies in any of the coordinate planes. For example, let P be a point such that $y_0 = 0$ and $x_0, z_0 \neq 0$. Equation (27) would force y_1 to be zero. With z_1 determined by Equation (28), it is possible that the point $(x_1, 0, z_1)$ will not lie on the surface of the ellipsoid.

Rather than modify the above procedure for cases where P lies in any of the coordinate planes, the problem will be analyzed by the method of Lagrange multipliers.

METHOD OF LAGRANGE MULTIPLIERS

Let the square of the distance between P and any point (x,y,z) on G, Equation (1), be given as

$$D = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2. \tag{31}$$

Define the function, f, as

$$f = D - \lambda G, \tag{32}$$

where λ is a Lagrange multiplier.

The solutions of the equations

$$\frac{\partial f}{\partial x} = 0,$$

$$\frac{\partial f}{\partial y} = 0, \tag{33}$$

$$\frac{\partial f}{\partial z} = 0,$$

are, respectively,

$$x(1 - \frac{\lambda}{a^2}) = x_0,$$

$$y(1 - \frac{\lambda}{b^2}) = y_0, \tag{34}$$

$$z(1 - \frac{\lambda}{c^2}) = z_0.$$

The set of Equations (34) and Equations (1) and (30) are used for determining Q and d when P is in any of the coordinate planes, as illustrated in the example that follows.

Given $P(x_0,0,0)$ and $x_0 \neq 0$. Let a, b, and c be distinct. Equations (34) take the form

$$x(1 - \frac{\lambda}{a^2}) = x_0, \quad \lambda \neq a^2,$$

$$y(1 - \frac{\lambda}{b^2}) = 0, \tag{35}$$

$$z(1 - \frac{\lambda}{c^2}) = 0.$$

If $\lambda \neq b^2, c^2$, Equations (35) yield

$$y = z = 0,$$

$$x = \frac{a^2 x_0}{a^2 - \lambda} \tag{36}$$

Equations (1) and (36) yield

$$\lambda = a^2 \pm a x_0$$

and thus $x = \pm a$. Consequently, a possible choice for Q is $(\pm a,0,0)$.

If $\lambda = b^2 \neq c^2$, Equations (35) yield

$$z = 0,$$

$$x = \frac{a^2 x_0}{a^2 - b^2}.$$

Substitution of the above information into Equation (1) yields

$$y = \pm b\left[1 - \frac{a^2x_0^2}{(a^2 - b^2)^2}\right]^{\frac{1}{2}}, \quad \left|\frac{ax_0}{a^2 - b^2}\right| < 1.$$

Thus, a possible choice for Q is

$$\left(\frac{a^2x_0}{a^2 - b^2}, \pm b\left[1 - \frac{a^2x_0^2}{(a^2 - b^2)^2}\right]^{\frac{1}{2}}, 0\right), \left|\frac{ax_0}{a^2 - b^2}\right| < 1.$$

If $\lambda = c^2 \neq b^2$, Equations (35) give

$$y = 0,$$

$$x = \frac{a^2x_0}{a^2 - c^2},$$

which, when substituted into Equation (1), give

$$z = \pm c\left[1 - \frac{a^2x_0^2}{(a^2 - c^2)^2}\right]^{\frac{1}{2}}, \quad \left|\frac{ax_0}{a^2 - c^2}\right| < 1.$$

Thus, Q could be the point

$$\left(\frac{a^2x_0}{a^2 - c^2}, 0, \pm c\left[1 - \frac{a^2x_0^2}{(a^2 - c^2)^2}\right]^{\frac{1}{2}}\right), \left|\frac{ax_0}{a^2 - c^2}\right| < 1.$$

In summary, if P is $(x_0,0,0)$, $x_0 \neq 0$, and a, b, and c are distinct, the candidates for Q are the following:

1. $(\pm a,0,0)$.

2. $$\left(\frac{a^2x_0}{a^2 - b^2},\ \pm b\left[1 - \frac{a^2x_0^2}{(a^2 - b^2)^2}\right]^{\frac{1}{2}},\ 0\right), \left|\frac{ax_0}{a^2 - b^2}\right| < 1.$$

3. $$\left(\frac{a^2x_0}{a^2 - c^2},\ 0,\ \pm c\left[1 - \frac{a^2x_0^2}{(a^2 - c^2)^2}\right]^{\frac{1}{2}}\right), \left|\frac{ax_0}{a^2 - c^2}\right| < 1.$$

Point Q is selected by the substitution of the above coordinates into Equation (30) and by examining the resulting values of d for a minimum.

Working with the basic Equations (34), we may determine possible candidates for Q when P is in any of the coordinate planes. A summary of possible coordinates for Q for various positions of P in the coordinate planes are given in Table I in Appendix I.

If several sets of coordinates for Q are listed for a particular case in Table I, Equation (30) is used to determine which set yields the minimum value for d; however, several cases exist where there could be an infinite number of possible positions for Q. Case 2, for example, shows that Q could lie on a circle located in the x,y-plane. By selecting any point on this circle, we can calculate d.

Several cases, such as Case 21, require that roots of quartics be determined. Again, the reader has the option to use his favorite scheme to obtain these roots.

OTHER METHODS

The dynamic programming technique and the method of Lagrange multipliers were used to find approximate solutions to the problem of the distance between a point and an ellipsoid[6]. The dynamic programming approach considered the axes of the ellipsoid to be parallel to the coordinate axes; the Lagrange multiplier approach considered the case where these axes did not necessarily have to be parallel.

It is not clear from the dynamic programming solution as to how situations such as, for example, Cases 2 to 5 in Appendix I, where an

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6. A. Celmins and W. Sacco, "Numerical Computations of the Distance from a Point to an Ellipsoid," Ballistic Research Laboratories Report No. 1328, 1966. (AD #641011)

infinite number of points, Q, solve the problem, are resolved. The computer algorithm, developed from the Lagrange multiplier method, considered P to be outside the ellipsoid; otherwise, the algorithm became too complicated.

This report presents exact solutions when possible; otherwise, the solutions are obtained by finding the roots of 6-th or 4-th degree polynomials in a known domain for the roots. For a given accuracy, a computer solution of the approach presented in this report should be faster than those in Reference 6. Furthermore, since P can be located inside or outside of the ellipsoid, all requirements for the application of this study to systems analysis have been satisfied.

SUMMARY

We wish to determine the shortest distance between a fixed point $P(x_0,y_0,z_0)$ and an ellipsoid. $Q(x_1,y_1,z_1)$ is defined to be on this surface such that the distance between P and Q is the minimum. Translation and rotation transformations should be used on the system consisting of a fixed point and an ellipsoid to describe the ellipsoid in standard form, Equation (1). The real roots of Equation (29), such that $|x_1| < a$, with their corresponding components, given by Equations (27) and (28), provide possible values for Q if P does not lie in any of the coordinate planes. If P does lie in one or more coordinate planes, Appendix I provides possible coordinates for Q. In either case, the final choice of coordinates for Q are determined by finding the minimum value for d by Equation (30).

APPENDIX I

TABLE I

POSSIBLE COORDINATES FOR $Q(x_1,y_1,z_1)$ WITH $P(x_o,y_o,z_o)$ IN THE COORDINATE PLANES

Case	$P(x_o,y_o,z_o)$	a,b,c	x_1	y_1	z_1	Restrictions and Comments
1.	(0,0,0)	distinct	$\pm a$	0	0	
			0	$\pm b$	0	
			0	0	$\pm c$	
2.	(0,0,0)	$a=b\neq c$	0	0	$\pm c$	
			*	*	0	* $x_1^2 + y_1^2 = a^2$, circle
3.	(0,0,0)	$b=c\neq a$	$\pm a$	0	0	
			0	*	*	* $y_1^2 + z_1^2 = b^2$, circle
4.	(0,0,0)	$c=a\neq b$	0	$\pm b$	0	
			*	*	0	* $z_1^2 + x_1^2 = c^2$, circle
5.	(0,0,0)	a=b=c	*	*	*	* $x_1^2 + y_1^2 + z_1^2 = a^2$, sphere
6.	$(x_o,0,0)$	distinct	$\pm a$	0	0	
			$\frac{a^2 x_o}{a^2 - b^2}$	$\pm b\left[1 - \frac{x_1^2}{a^2}\right]^{\frac{1}{2}}$	0	$\lvert x_1 \rvert \leq a$
			$\frac{a^2 x_o}{a^2 - c^2}$	0	$\pm c\left[1 - \frac{x_1^2}{a^2}\right]^{\frac{1}{2}}$	$\lvert x_1 \rvert \leq a$
7.	$(0,y_o,0)$	distinct	0	$\pm b$	0	
			0	$\frac{b^2 y_o}{b^2 - c^2}$	$\pm c\left[1 - \frac{y_1^2}{b^2}\right]^{\frac{1}{2}}$	$\lvert y_1 \rvert \leq b$
			$\pm a\left[1 - \frac{y_1^2}{b^2}\right]^{\frac{1}{2}}$	$\frac{b^2 y_o}{b^2 - a^2}$	0	$\lvert y_1 \rvert \leq b$

TABLE I (con't)

Case	$P(x_o,y_o,z_o)$	a,b,c	x_1	y_1	z_1	Restrictions and Comments
8.	$(0,0,z_o)$	distinct	0	0	$\pm c$	
			$\pm a\left[1 - \frac{z_1^2}{c^2}\right]^{\frac{1}{2}}$	0	$\frac{c^2 z_o}{c^2 - a^2}$	$\lvert z_1 \rvert \le c$
			0	$\pm b\left[1 - \frac{z_1^2}{c^2}\right]^{\frac{1}{2}}$	$\frac{c^2 z_o}{c^2 - b^2}$	$\lvert z_1 \rvert \le c$
9.	$(x_o,0,0)$	a=b≠c	$\pm a$	0	0	
			$\frac{a^2 x_o}{a^2 - c^2}$	0	$\pm c\left[1 - \frac{x_1^2}{a^2}\right]^{\frac{1}{2}}$	$\lvert x_1 \rvert \le a$
10.	$(0,y_o,0)$	b=c≠a	0	$\pm b$	0 *	
			$\pm a\left[1 - \frac{y_1^2}{b^2}\right]^{\frac{1}{2}}$	$\frac{b^2 y_o}{b^2 - a^2}$	0	$\lvert y_1 \rvert \le b$
11.	$(0,0,z_o)$	c=a≠b	0	0	$\pm c$	
			0	$\pm b\left[1 - \frac{z_1^2}{c^2}\right]^{\frac{1}{2}}$	$\frac{c^2 z_o}{c^2 - b^2}$	$\lvert z_1 \rvert \le c$
12.	$(x_o,0,0)$	b=c≠a	$\pm a$	0	0	
			$\frac{a^2 x_o}{a^2 - b^2}$	*	*	* $y_1^2 + z_1^2 = b^2\left[1 - \frac{x_1^2}{a^2}\right]$, circle if $\lvert x_1 \rvert < a$
13.	$(0,y_o,0)$	c=a≠b	0	$\pm b$	0	
			*	$\frac{b^2 y_o}{b^2 - c^2}$	*	* $z_1^2 + x_1^2 = c^2\left[1 - \frac{y_1^2}{b^2}\right]$, circle if $\lvert y_1 \rvert < b$
14.	$(0,0,z_o)$	a=b≠c	0	0	$\pm c$	
			*	*	$\frac{c^2 z_o}{c^2 - a^2}$	* $x_1^2 + y_1^2 = a^2\left[1 - \frac{z_1^2}{c^2}\right]$, circle if $\lvert z_1 \rvert < c$

TABLE I (con't)

Case	$P(x_o,y_o,z_o)$	a,b,c	x_1	y_1	z_1	Restrictions and Comments
15	$(x_o,0,0)$	$c=a\neq b$	$\pm a$	0	0	
			$\frac{a^2x_o}{a^2-b^2}$	$\pm b\left[1-\frac{x_1^2}{a^2}\right]^{\frac{1}{2}}$	0	$\lvert x_1\rvert \leq a$
16.	$(0,y_o,0)$	$a=b\neq c$	0	$\pm b$	0	
			0	$\frac{b^2y_o}{b^2-c^2}$	$\pm c\left[1-\frac{y_1^2}{b^2}\right]^{\frac{1}{2}}$	$\lvert y_1\rvert \leq b$
17.	$(0,0,z_o)$	$b=c\neq a$	0	0	$\pm c$	
			$\pm a\left[1-\frac{z_1^2}{c^2}\right]^{\frac{1}{2}}$	0	$\frac{c^2z_o}{c^2-a^2}$	$\lvert z_1\rvert \leq c$
18.	$(x_o,0,0)$	$a=b=c$	$\pm a$	0	0	
19.	$(0,y_o,0)$	$a=b=c$	0	$\pm b$	0	
20.	$(0,0,z_o)$	$a=b=c$	0	0	$\pm c$	
21.	$(x_o,y_o,0)$	distinct	*	$\pm b\left[1-\frac{x_1^2}{a^2}\right]^{\frac{1}{2}}$	0	* $(b^2-a^2)^2 x_1^4 + 2a^2x_o(b^2-a^2)\,x_1^3$ $+ a^2\left[a^2x_o^2 + b^2y_o^2 - (b^2-a^2)^2\right]x_1^2$ $-2a^4x_o(b^2-a^2)\,x_1 - a^6x_o^2 = 0, \quad \lvert x_1\rvert \leq a$
			$\frac{a^2x_o}{a^2-c^2}$	$\frac{b^2y_o}{b^2-c^2}$	$\pm c\left[1-\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}\right]^{\frac{1}{2}}$	$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} \leq 1$

TABLE I (con't)

Case	$P(x_o,y_o,z_o)$	a,b,c	x_1	y_1	z_1	Restrictions and Comments
22.	$(x_o,0,z_o)$	distinct	$\pm a\left[1 - \frac{z_1^2}{c^2}\right]^{\frac{1}{2}}$	0	*	* $(a^2-c^2)^2 z_1^4 + 2c^2 z_o(a^2-c^2) z_1^3 + c^2\left[c^2 z_o^2 + a^2 x_o^2 - (a^2-c^2)^2\right] z_1^2 - 2c^4 z_o(a^2-c^2) z_1 - c^6 z_o^2 = 0, \quad \lvert z_1\rvert \le c$
			$\frac{a^2 x_o}{a^2 - b^2}$	$\pm b\left[1 - \frac{x_1^2}{a^2} - \frac{z_1^2}{c^2}\right]^{\frac{1}{2}}$	$\frac{c^2 z_o}{c^2 - b^2}$	$\frac{x_1^2}{a^2} + \frac{z_1^2}{c^2} \le 1$
23.	$(0,y_o,z_o)$	distinct	0	*	$\pm c\left[1 - \frac{y_1^2}{b^2}\right]^{\frac{1}{2}}$	* $(c^2-b^2)^2 y_1^4 + 2b^2 y_o(c^2-b^2) y_1^3 + b^2\left[b^2 y_o^2 + c^2 z_o^2 - (c^2-b^2)^2\right] y_1^2 - 2b^4 y_o(c^2-b^2) y_1 - b^6 y_o^2 = 0, \quad \lvert y_1\rvert \le b$
			$\pm a\left[1 - \frac{y_1^2}{b^2} - \frac{z_1^2}{c^2}\right]^{\frac{1}{2}}$	$\frac{b^2 y_o}{b^2 - a^2}$	$\frac{c^2 z_o}{c^2 - a^2}$	$\frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} \le 1$
24.	$(x_o,y_o,0)$	$a=b\neq c$	$\pm \frac{x_o a}{\left[x_o^2 + y_o^2\right]^{\frac{1}{2}}}$	$\pm \frac{y_o a}{\left[x_o^2 + y_o^2\right]^{\frac{1}{2}}}$	0	
			$\frac{a^2 x_o}{a^2 - c^2}$	$\frac{a^2 y_o}{a^2 - c^2}$	$\pm \frac{c}{a}\left[a^2 - x_1^2 - y_1^2\right]^{\frac{1}{2}}$	$x_1^2 + y_1^2 \le a^2$
25.	$(0,y_o,z_o)$	$b=c\neq a$	0	$\pm \frac{y_o b}{\left[y_o^2 + z_o^2\right]^{\frac{1}{2}}}$	$\pm \frac{z_o b}{\left[y_o^2 + z_o^2\right]^{\frac{1}{2}}}$	
			$\pm \frac{a}{b}\left[b^2 - y_1^2 - z_1^2\right]^{\frac{1}{2}}$	$\frac{b^2 y_o}{b^2 - a^2}$	$\frac{b^2 z_o}{b^2 - a^2}$	$y_1^2 + z_1^2 \le b^2$

TABLE I (con't)

Case	$P(x_o,y_o,z_o)$	a,b,c	x_1	y_1	z_1	Restrictions and Comments
26.	$(x_o,0,z_o)$	$c=a\neq b$	$\pm \dfrac{x_o c}{\left[z_o^2 + x_o^2\right]^{\frac{1}{2}}}$	0	$\pm \dfrac{z_o c}{\left[z_o^2 + x_o^2\right]^{\frac{1}{2}}}$	
			$\dfrac{c^2 x_o}{c^2 - b^2}$	$\pm \dfrac{b}{c}\left[c^2 - z_1^2 - x_1^2\right]^{\frac{1}{2}}$	$\dfrac{c^2 z_o}{c^2 - b^2}$	$z_1^2 + x_1^2 \leq c^2$
27.	$(x_o,y_o,0)$	$b=c\neq a$	*	$\dfrac{b^2 y_o x_1}{b^2 x_1 - a^2(x_1 - x_o)}$	0	* $(b^2-a^2)^2 x_1^4 + 2a^2 x_o (b^2-a^2) x_1^3$ $+ a^2\left[a^2 x_o^2 + b^2 y_o^2 - (b^2 - a^2)^2\right] x_1^2$ $- 2a^4 x_o\left[b^2 - a^2\right] x_1 - a^6 x_o^2 = 0$
28.	$(0,y_o,z_o)$	$c=a\neq b$	0	*	$\dfrac{c^2 z_o y_1}{c^2 y_1 - b^2(y_1 - y_o)}$	* $(c^2-b^2)^2 y_1^4 + 2b^2 y_o (c^2-b^2) y_1^3$ $+ b^2\left[b^2 y_o^2 + c^2 z_o^2 - (c^2-b^2)^2\right] y_1^2$ $- 2b^4 y_o (c^2 - b^2) y_1 - b^6 y_o^2 = 0$
29.	$(x_o,0,z_o)$	$a=b\neq c$	$\dfrac{a^2 x_o z_1}{a^2 z_1 - c^2(z_1 - z_o)}$	0	*	* $(a^2-c^2)^2 z_1^4 + 2c^2 z_o (a^2-c^2) z_1^3$ $+ c^2\left[c^2 z_o^2 + a^2 x_o^2 - (a^2-c^2)^2\right] z_1^2$ $- 2c^4 z_o (a^2 - c^2) z_1 - c^6 z_o^2 = 0$
30.	$(x_o,y_o,0)$	$c=a\neq b$	*	*	*	* Same as Case 27
31.	$(0,y_o,z_o)$	$a=b\neq c$	*	*	*	* Same as Case 28
32.	$(x_o,0,z_o)$	$b=c\neq a$	*	*	*	* Same as Case 29
33.	$(x_o,y_o,0)$	$a=b=c$	$\pm \dfrac{x_o a}{\left[x_o^2 + y_o^2\right]^{\frac{1}{2}}}$	$\pm \dfrac{y_o a}{\left[x_o^2 + y_o^2\right]^{\frac{1}{2}}}$	0	

TABLE I (con't)

Case	$P(x_o,y_o,z_o)$	a,b,c	x_1	y_1	z_1	Restrictions and Comments
34.	$(0,y_o,z_o)$	a=b=c	0	$\pm \frac{y_o b}{[y_o^2 + z_o^2]^{\frac{1}{2}}}$	$\pm \frac{z_o b}{[y_o^2 + z_o^2]^{\frac{1}{2}}}$	
35.	$(x_o,0,z_o)$	a=b=c	$\pm \frac{x_o c}{[z_o^2 + x_o^2]^{\frac{1}{2}}}$	0	$\pm \frac{z_o c}{[z_o^2 + x_o^2]^{\frac{1}{2}}}$	

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